The Exterior Plateau Problem in Higher Codimension

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Abstract

We prove existence theorems for two-dimensional non-compact complete minimal surfaces in $\mathbb{R}^n$ of annular type which span a given contour and have a finite total curvature end and prescribed asymptotical behaviour. For arbitrary rectifiable Jordan curves we show the existence of such surfaces with a flat end, i. e. within bounded distance from a 2-plane. For more restricted classes of curves we prove the existence of minimal surfaces with higher multiplicity flat ends as well as of surfaces with polynomial type non-flat ends.

1 Introduction

Let Γ be a rectifiable Jordan curve in $\mathbb{R}^n$. The exterior Plateau problem for Γ asks for a complete non-compact minimal surface spanning Γ. While geometric measure theory may easily provide the existence of solutions of unspecified topological type to this problem, we in this paper are interested in finding surfaces of annular type which moreover have a specified asymptotical behaviour at infinity. Whereas embedded finite total curvature ends of minimal surfaces in $\mathbb{R}^3$ must be asymptotic to a plane or to a half catenoid, there is a large variety of different such ends in $\mathbb{R}^n$ as soon as $n \geq 4$: algebraic complex curves in a complex subspace of $\mathbb{R}^n$ yield plenty of examples. This makes the higher codimensional Plateau problem even more interesting than the corresponding problem in $\mathbb{R}^3$ which was investigated by R. Ye and one of the present authors [T-Y]. Though in higher codimension one may construct many candidates of solutions of different asymptotic behaviour, it seems to be quite difficult to actually verify their behaviour at infinity, apart from the simplest case of a flat end. This difficulty is due to the lack of the usual comparison principle for codimension one mimimal surfaces. Instead we must rely on a more general barrier principle for minimal surfaces of arbitrary codimension which we established in a previous paper [J-T]. So far, our supply of useful and non-trivial barriers is however rather limited, due to their cumbersome construction.

The plan of the paper is as follows. In section 2 we first prove an existence result for compact minimal annuli which is based on Morgan's characterization of area minimizing sums of two

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2-planes in $\mathbb{R}^n$ [M]. We then apply this to construct families of expanding compact minimizing annuli spanning a fixed "inner" Jordan curve Γ and an "outer" Jordan curve from a family Γ_R, $R \geq R_0$, which is obtained as the intersection of a model surface M_0 with spheres of radius R. Here we assume that M_0 is a complete simply connected, area-minimizing, finite total curvature minimal surface with an embedded end and that the asymptotical tangent plane of M_0 is not orthogonal to the affine subspace of minimal dimension containing Γ. Subsequently, we show that there are sequences $R_k \to +\infty (k \to \infty)$ such that the minimizing annuli spanning $\Gamma \cup \Gamma_{R_k}$ converge to a limit surface when $k \to \infty$. In section 3 we investigate the asymptotical behaviour of the limit surfaces and show that they are complete and have a finite total curvature end. In the case that M_0 is a plane we prove that the limit surfaces have a flat end of geometric index 1, the asymptotic tangent plane having the direction of M_0. In section 4, we apply our barrier principle [J-T] and the concept of incompressible surfaces to establish the existence of complete finite total curvature surfaces with a flat end of geometric index $m > 1$ spanning the Jordan curve Γ provided that Γ is contained in the non-simply connected domain bounded by one of our catenoidal barriers and has the winding number m. Finally, in section 5 we construct a barrier around the end of the model surface $M_0 = \{(z, w) \in \mathbb{C} \times \mathbb{C} | w = z^m\}, m \in \mathbb{N}$, which is contained in a distance 1 neighborhood of M_0. This immediately leads to an existence theorem for solutions to Plateau's problem asymptotic to this M_0.

Some of the arguments in the sections 2 and 3 of this paper are identical with those in the codimension 1 case treated in [T-Y], some arguments are essentially different. For instance, the curvature estimates used in [T-Y] are not available in codimension > 1. We present the complete proofs in order to make the paper better readable.

2 The Expanding Minimal Annuli

The basis of our construction is Morgan's characterization of the singularity structure of two-dimensional area minimize surfaces in $\mathbb{R}^n$ [M]. We use this characterization in the following Lemma to verify Douglas' condition for the existence of least area annuli spanning a given pair of Jordan curves in $\mathbb{R}^n$ [D].

Let us introduce some notation:

$$D_R := \{u = (u_1, u_2) \in \mathbb{R}^2 | \mid u \mid \leq R\}, \, R > 0,$$

$$A_R := D_R \setminus \text{int}(D_1) \quad R > 1,$$

$$B_\rho(p) := \{x \in \mathbb{R}^n \big| |x - p| \leq \rho\}, \, p \in \mathbb{R}^n, \rho > 0.$$

Lemma 2.1 *Let M_1, $M_2 \subset \mathbb{R}^n$ be classical disc type least area surfaces spanning the rectifiable Jordan curves Γ_1 and Γ_2, respectively, i.e. M_1 and M_2 are the images of continuous maps $x_k : D_1 \to \mathbb{R}^n, k = 1, 2$, where $x_k | \partial D_1$ parameterizes Γ_k, x_k is a harmonic conformal branched immersion in the interior of D_1, and x_k minimizes area among all maps $y \in C^0(D_1, \mathbb{R}^n) \cap$*

2

$C^{0,1}(\mathring{D}_1, \mathbb{R}^n)$ *such that* $y|\partial D_1$ *parameterizes* Γ_k *weakly monotonically. We assume furthermore that there are points* $w_i \in \mathring{D}_1, k = 1, 2$, *such that* w_k *is regular for* $x_k, x_1(w_1) = x_2(w_2)$, *and the tangent spaces* T_k *of* x_k *at* w_k *are not mutually orthogonal. Then there is an orientation* $\mathcal{O}$ *of* $\Gamma_1 \cup \Gamma_2$ *and a number* $\delta > 0$ *such that*

$$(2.1) \qquad\qquad a(\Gamma_1 \cup \Gamma_2, \mathcal{O}) \leq area(M_1) + area(M_2) - \delta,$$

where $a(\Gamma_1 \cup \Gamma_2, \mathcal{O})$ *denotes the infinium of the area of all mappings* $y \in C^0(A_2, \mathbb{R}^n) \cap$ $C^{0,1}(\mathring{A}_2, \mathbb{R}^n)$ *such that* $y|\partial A_2$ *parameterizes* $\Gamma_1 \cup \Gamma_2$ *coherent with the orientation* $\mathcal{O}$. *Hence, it follows from Douglas' theorem [D] that* $\Gamma_1 \cup \Gamma_2$ *with the chosen orientation* $\mathcal{O}$ *spans a classical least area annulus, i.e. there are a number* $r > 1$ *and a harmonic, conformal, branched immersion* x *of* $\mathring{A}_r$ *into* $\mathbb{R}^n$ *which is continuous on* A_r *and such that* $x|\partial A_r$ *parameterizes* $\Gamma_1 \cup \Gamma_2$ *coherent with* $\mathcal{O}$ *and* $area(x) = a(\Gamma_1 \cup \Gamma_2, \mathcal{O})$.

Proof: We may assume that $x_1(w_1) = x_2(w_2) = 0$, the origin of our coordinate system.

Let us denote by $\Delta_k(\rho) \subset T_k$ the discs of radius ρ and center 0. Since by assumption T_1 and T_2 are not orthogonal, it follows from Corollary 6 of [M] that the discs $\Delta_1(1)$ and $\Delta_2(1)$ can be oriented in such a way that $\Delta_1(1) + \Delta_2(1)$ is not area minimizing in the class of oriented surfaces. We choose such an orientation $\mathcal{O}$ of $\Delta_1(1)$ and $\Delta_2(1)$ and, if necessary, we reorient the surfaces x_1 and x_2 in such a way that the induced orientation of the tangent spaces T_1 and T_2 coincides with $\mathcal{O}$. Beyond the statement of Morgan's Corollary 6 we need the following more precise information from the proof of Theorem 2 in the same paper: there is a smooth annulus type surface C with $\partial C = \partial \Delta_1(1) + \partial \Delta_2(1)$ and $area(C) < 2\pi$. Since x_k is an embedding near w_k for $k = 1, 2$ there are neighborhoods U_k of w_k and $\rho_o > 0$ such that $x_k(U_k)$ has a one-to-one orthogonal projection onto $\Delta_k(\rho_o) \subset T_k$. Let us denote by $M_k(\rho)$ the piece of $x_k(U_k)$ which projects onto $\Delta_k(\rho)$; $k = 1, 2$; $0 < \rho \leq \rho_o$. By the smoothness of x_k the vertical height of $M_k(\rho)$ over $\Delta_k(\rho)$ is estimated by const.ρ^2 and we may therefore construct cylindrical annuli $C_k(\rho)$ with $\partial C_k(\rho) = \partial M_k(\rho) - \partial \Delta_k(\rho)$ and $area(C_k(\rho)) \leq$ const.ρ^3. It follows that $S_\rho := C_1(\rho) + \rho C + C_2(\rho)$ is an annulus with boundary $\partial S_\rho = \partial M_1(\rho) + \partial M_2(\rho)$ and $area(S_\rho) \leq \rho^2 area(C_1(1)) +$ const.ρ^3. Since $area(C_1(1)) < 2\pi$ we may find $\rho > 0$ and $\delta > 0$ such that $area(S_\rho) \leq 2\pi\rho^2 - \delta$ and hence

$$(2.2) \qquad\qquad area(S_\rho) \leq area(M_1(\rho)) + area(M_2(\rho)) - \delta.$$

We may now delete the discs $M_k(\rho)$ from M_k and join the resulting annuli $M_k - M_k(\rho)$ by S_ρ to obtain an annulus M_ρ with $\partial M_\rho = \Gamma_1 \cup \Gamma_2$ and, as follows immediately from (2.2),

$$area(M_\rho) \leq area(M_1) + area(M_2) - \delta.$$

Remark 2.2 *As the above proof shows, the number* δ *in the Douglas condition(2.1) depends only on the restriction of the surfaces* x_k *onto some neighborhood of* w_k $(k = 1, 2)$.

We shall now apply Lemma 2.1 in the situation when M_1 is a fixed least area disc M spanning the given Jordan curve Γ and M_2 is the intersection of a model surface M_0 with a ball of radius R, denoted by M_0^R. We assume the following about M_0: M_0 is a complete, simply connected, finite total curvature minimal surface with an embedded end and all compact subdiscs of M_0 are area-minimizing among disc type surfaces. Complex algebraic curves in some complex subspace of $\mathbb{R}^n$ yield plenty of examples of such model surfaces.

It follows from the above properties of M_0 that M_0 possesses a limit tangent space at infinity which we denote by T.

Let then $\Gamma \subset \mathbb{R}^n, n \geq 4$, be a rectifiable Jordan curve contained in the unit ball of $\mathbb{R}^n$, and let $V \subset \mathbb{R}^n$ be the affine subspace of $\mathbb{R}^n$ of minimal dimension containing Γ. Let M denote a classical least area disc spanning Γ. By the convex hull property of minimal surfaces, M is contained in V and by the minimality of $\dim V$ the tangent spaces of M generate the tangent space of V. Hence, if the asymptotic tangent space of T of M_0 is not orthogonal to V, there is an open set of points on M whose tangent spaces are not orthogonal to T. We may therefore choose a regular point p on M and a regular point q on M_0 such that their tangent spaces are not orthogonal. After a translation of M_0 we may assume that $p = q$. We finally observe that $\Gamma_R := \partial M_0^R$ is a smooth Jordan curve for $R \geq R_0 > 1$ since the end of M_0 is embedded by assumption. From Lemma 2.1 and Remark 2.2 we therefore obtain

Lemma 2.3 *Let $\Gamma \subset B_1(0)$ be a rectifiable Jordan curve, and M_0 as described above, and assume that the subspace of minimal dimension containing Γ is not orthogonal to T, the asymptotic tangent space of M_0. Then, after translating and possibly reorienting M_0 there exist, for any $R > R_0$, classical least area annuli M_R spanning $\Gamma \cup \Gamma_R$. The annuli M_R are given by harmonic, conformal, branched immersions $x_R : \overset{\circ}{A}_{r(R)} \to \mathbb{R}^n$, where x_R is continuous on $A_{r(R)}$ and $x_R|\partial A_{r(R)}$ maps $\partial A_{r(R)}$ topologically onto $\Gamma \cup \Gamma_R$, coherent with the chosen orientation of $\Gamma \cup \Gamma_R$. Moreover, there is a number $\delta > 0$ such that the inequalities*

$$(2.3) \qquad\qquad area(M_R) \leq area(M) + area(M_o^R) - \delta$$

hold for all $R > R_0$, where M is a least area disc spanning Γ.

By a well know cut-and-paste argument [T-T] one infers from the uniform Douglas-condition (2.3) a uniform "condition of cohesion".

Corollary 2.4 *There is a number $\varepsilon > 0$ such that the length of the images under any x_R of all homotopically non-trivial loops in $A_{r(R)}$ is bounded below by ε. One may take any $\varepsilon < \sqrt{\delta}$.*

Now we are going to establish uniform (with respect to R) local area estimates for the surfaces M_R. These follow easily from the monotonicity formula for minimal surfaces with boundary [E-W-W]: let M be any minimal surface in $\mathbb{R}^n$ with C^1 boundary ∂M and for $p \in \mathbb{R}^n$ and $\rho > 0$ let us define

4

$$M_\rho := M \cap B_\rho(p), \quad (\partial M)_\rho := \partial M \cap B_\rho(p), \quad d(x) := |x - p|.$$

Then the following formula holds:

$$\frac{\partial}{\partial \rho} \left(\frac{\mathrm{area}(M_\rho)}{\rho^2} \right) \geq -\frac{1}{\rho^3} \int_{(\partial M)_\rho} d\langle \nabla d, \nu \rangle ds,$$

where ν denotes the unit exterior surface normal along ∂M. If $(\partial M)_\rho$ is contained in $B_r(p)$ we obtain the estimate

$$\frac{\partial}{\partial \rho} \left(\frac{\mathrm{area}(M_\rho)}{\rho^2} \right) \geq -\frac{r}{\rho^3} \mathrm{length}(\partial M)_\rho.$$

Let us apply this formula to our annuli with $p = 0$ and $1 < \rho < R$ such that $(\partial M)_\rho = \Gamma$ and consequently $r \leq 1$. By integration we obtain

(2.4)
$$R^{-2}\mathrm{area}(M_R) - \rho^{-2}\mathrm{area}((M_R)_\rho) \geq \left(\frac{1}{2}R^{-2} - \frac{1}{2}\rho^{-2} \right) \mathrm{length}(\Gamma)$$

It is well known, that M_0 has quadratic area growth, i.e. there exist positive numbers a and m such that,

$$\mathrm{area}(M_0^R) \leq a + m\,\pi R^2\,.$$

Combining this with (2.3) and (2.4) we obtain

Lemma 2.5 *The annuli M_R fulfill the estimate*

$$area((M_R)_\rho) \leq \gamma + m\pi\rho^2, \quad 1 \leq \rho \leq R,$$

with $\gamma := area(M) + 1/2 length(\Gamma) + a$.

Completely analogous to the proceeding in [T-Y] we may now control the conformal parameterization $x_R : A_{r(R)} \to \mathbb{R}^n$ of M_R. Because of the conformality the area of x_R over any domain G coincides with the Dirichlet energy.

$$E(x_R, G) := \frac{1}{2} \iint_G \left(\left| \frac{\partial}{\partial u_1} \right|^2 + \left| \frac{\partial}{\partial u_2} \right|^2 \right) du_1 du_2$$

We introduce the following subdomains of the parameter domain $A_{r(R)}$:

$$\Omega(x_R, \rho) : = (x_R)^{-1}(B_\rho),$$
$$A(x_R, \rho) : = \text{the component of } \Omega(x_R, \rho) \text{ containing } \partial D_1 = S^1$$

Lemma 2.6

> (i) $dist(\partial A(x_R, \rho) \setminus S^1, \partial A(x_R, 2\rho) \setminus S^1) \geq \exp(-4\pi(\gamma + m\pi)),\ 1 < \rho < R/2$
> (ii) $dist(O, \partial A(x_R, \rho)) \geq c \ln \rho$ *for some positive constant* $c,\ \ 1 < \rho < R$
> (iii) $r(R) \geq c \ln r$

Proof: Let us choose a point $z \in \partial A(x_R, \rho) \setminus S^1$ such that

$$\text{dist}(z, \partial A(x_R, 2\rho) \setminus S^1) = r_o := \text{dist}(\partial A(x_R, \rho) \setminus S^1, \partial A(x_R, 2\rho) \setminus S^1).$$

We need only consider the case when $r_o < 1$. Let us define $D_r(z) := \{u \in \mathbb{R}^2 \mid |u - z| \leq r\}$ and $\partial^* D_r(z) := \partial D_r(z) \cap A(x_R, 2\rho)$. According to the Lemma of Courant-Lebesgue [C, p. 101] there exists $r_1 \in (r_o, \sqrt{r_o})$ such that

(2.5)
$$\left(\int_{\partial^* D_{r_1(z)}} \left| \frac{\partial x_R}{\partial \theta} \right| d\theta \right)^2 \leq \frac{4\pi}{-\ln r_o} E(x_R, A(x_R, 2\rho))$$

$$\leq -\frac{4\pi}{\ln r_o} \text{area}((M_R)_{2\rho}).$$

Since $r_1 \geq r_o$ and by the definition of $A(x_R, 2\rho)$ there must be an arc in $\partial^* D_{r_1}(z)$ connecting $\partial A(x_R, 2\rho)$ with $\Omega(x_R, \rho)$. Hence the left hand side of (2.5) has the lower bound ρ^2 and Lemma 2.5 immediately gives (i). Assertion (ii) is an easy consequence of (i)(cf. Lemma 2.4 of [T-Y]) and (iii) follows trivially from (ii).

Lemma 2.5 and Lemma 2.6 yield directly the

Corollary 2.7

> (i) $\sup_{A_r} |x_R| \leq e^{r/c}$
> (ii) $E(x_R, A_r) \leq \gamma + m\pi e^{2r/c},\ \ 1 < r < r(R).$

We may now prove the existence of limit surfaces to our family of least area annuli. We define $A := \{u \in \mathbb{R}^2 | \ |u| \geq 1\}$.

Proposition 2.8 *For every sequence $R_k \to +\infty (k \to \infty)$ there is a subsequence (again denoted by R_k) and a map $x \in C^0(A, \mathbb{R}^n) \cap C^\infty(\mathring{A}, \mathbb{R}^n)$ with the following properties:*

    (i) $x| \mathring{A}$ *is a harmonic, conformal branched immersion,*

    (ii) $x|\partial A$ *maps ∂A topologically onto Γ,*

   (iii) $x_{R_k} \to x(k \to \infty)$ *in $C^\infty(\mathring{A})$, in $C^0(A_r)$, and weakly in $W^{1,2}(A_r)$ for every $r > 1$.*

Proof: The existence of a subsequence of (R_k) and a limit map $x : \mathring{A} \to \mathbb{R}^n$ such that $x_{R_k} \to x$ in $C^\infty(\mathring{A})$ and weakly in $W^{1,2}(A_r)$ for every r follows immediately from Corollary 2.7. The map x must be harmonic and conformal and, unless it is constant, a branched immersion as a C^∞-limit of maps with these properties. From the local energy bound (Corollary 2.7) and the condition of cohesion (Corollary 2.4) we obtain by a well-know argument (cf. T-T) the uniform continuity of the sequence $(x_{R_k}|S^1)$. It follows that x extends as a continuous map on A and that x maps ∂A weakly monotonically onto Γ, in particular, x is not constant. Since x is conformal and harmonic, $x|\partial A$ must actually be a topological map.

3   The asymptotical behavior of the limit surfaces

In this section we investigate further the properties of a minimal surface $x : A \to \mathbb{R}^n$ which is obtained by the limiting process of Proposition 2.8. In particular, we would like to show that x has a properly immersed finite total curvature end. Finally, we prove our first main result, Theorem 3.5 about solutions with a flat end as announced in the introduction.

Lemma 3.1 $E(x, \Omega(x, \rho)) \leq \gamma + m\pi\rho^2, \ \ \rho > 1.$

Proof: Let C be any compact subset of $\mathring{\Omega}(x, \rho)$. Because of the convergence $x_{R_k} \to x, k \to \infty$, we have $C \subset \Omega(x_{R_k}, \rho)$ for sufficiently large k and Lemma 2.5 yields

$$\begin{aligned} E(x, C) &\leq \liminf_{k\to\infty} E(x_{R_k}, C) \\ &\leq \gamma + m\pi\rho^2. \end{aligned}$$

The assertion of the lemma follows.

Lemma 3.2 *Each component of $\Omega(x, \rho)$ is a bounded subset of A for any $\rho > 1$.*

Proof: (cf. [T-Y]) Let us first show that $\Omega(x, \rho) \neq A$ for any ρ, i.e. that x is not bounded. We consider the map $x^* : D_1^* \to \mathbb{R}^n, \ D_1^* := D_1 \setminus \{0\}, \ x^*(z) := x(1/z)$. If x were bounded, so were x^* and hence x^* had a removable singularity at the origin. It follows that the length of $x^*(\partial D_r) = x(\partial D_{1/r})$ converges to 0 when $r \to 0$, whence the length$(x_{R_k}(\partial D_\rho))$ becomes arbitrarily small for large k and ρ, contradicting Corollary 2.4.

Let us now assume that Ω_o is an unbounded component of $\Omega(x, \rho)$ for some ρ. Let us choose a regular value ρ' of $d \circ x = |x|$, $\rho' > \rho$, and let Ω' denote the component of $\Omega(x, \rho')$ containing Ω_o. Then Ω' is also unbounded and hence $\partial\Omega' \setminus S^1$ consists of finitely many properly embedded unbounded analytic curves. Let us again consider x^* instead of x and let Ω^* be the image of Ω' under the map $z \mapsto 1/z$. From Lemma 3.1 and the invariance of Dirichlet's integral under conformal maps it follows that $E(x^*, \Omega^*)$ is finite and we may hence apply the lemma of Courant-Lebesgue [C] to find a radius r, $0 < r < 1$, such that $\text{length}(x^*(\Omega^* \cap \partial D_r)) < \rho' - \rho$. Since any component of $\Omega^* \cap \partial D_r$ must meet $\partial\Omega^*$ we conclude that $|x^*| > \rho$ on $\Omega^* \cap \partial D_r$. Hence $\Omega' \cap A_{1/r}$ is a bounded set such that $|x| > \rho$ on $\partial(\Omega' \cap A_{1/r}) \setminus S^1$. We may choose r so small that $\Omega' \cap A_{1/r}$ contains some point of Ω_o. It follows then that $\Omega_o \subset \Omega' \cap A_{1/r}$ and hence Ω_o is bounded.

Proposition 3.3 *The map x is proper, i.e. $|x(u)| \to +\infty$ for $|u| \to +\infty$.*

Proof: Let $\rho > 1$ be arbitrarily chosen and let Ω_1, $\Omega_2, \ldots$ denote those components of $\Omega(x, 2\rho)$ which are different from $A(x, 2\rho)$ and contain some point $w_k \in \Omega_k$ with $|x(w_k)| < \rho$. Then $x(\Omega_k)$ is a disc type minimal surface which has no boundary in the interior of the ball $B_\rho(x(w_k))$. It follows from the monotonicity formula (cf. section 2) that $E(x, \Omega_k) \geq \pi\rho^2$. In view of Lemma 3.1 we conclude that there can exist only finitely many such Ω_k, say $\Omega_1, \ldots, \Omega_l$. On the complement of $A(x, 2\rho) \cup \Omega_1 \cup \ldots \cup \Omega_l$ we have the inequality $|x| \geq \rho$ and hence $\Omega(x, \rho) \subset A(x, 2\rho) \cup \Omega_1 \cup \ldots \cup \Omega_l$. The latter is a bounded set by Lemma 3.2 and hence $\Omega(x, \rho)$ is bounded.

Proposition 3.4 *For each $\rho > 1, x|A \setminus A_\rho$ is a finite total curvature minimal surface with only finitely many branch points; in particular, x has an immersed end.*

Proof: Branch points being isolated points, we may choose a radius $\sigma > \rho > 1$, σ arbitrarily large and ρ arbitrarily close to 1 such that x and x_{R_k} are immersed along $\partial D_\rho \cup \partial D_\sigma$ for $k \geq k_0$. Denoting by $O(x, u)$ the order of a branch point u of a minimal surface x and setting $O(x, u) = 0$ for regular points u, we obtain from the Gauss-Bonnet formula for branched minimal surfaces [D-H-K-W, Ch. 7.11]

$$2\pi \sum_{u \in A_\sigma \setminus A_\rho} O(x, u) - \iint_{A_\sigma \setminus A_\rho} K d\omega = \int_{\partial(A_\sigma \setminus A_\rho)} \kappa_g ds$$

where K and $d\omega$ denote the Gauss curvature and the area element of x and κ_g and ds the geodesic curvature and the arclength element of the boundary curves $x(\partial(A_\sigma \setminus A_\rho))$. By the smooth convergence of x_{R_k} to x on $\mathring{A}$ and again applying the Gauss-Bonnet formula on x_{R_k} we obtain

$$
\begin{aligned}
2\pi \sum_{u \in A_\sigma \backslash A_\rho} O(x,u) - \iint_{A_\sigma \backslash A_\rho} K d\omega &= \lim_{k\to\infty} \int_{\partial(A_\sigma \backslash A_\rho)} \kappa_g^k ds^k \\
&= \lim_{k\to\infty} \left(2\pi \sum_{u \in A_\sigma \backslash A_\rho} O(x^k,u) - \iint_{A_\sigma \backslash A_\rho} K^k d\omega^k \right) \\
&\leq \limsup_{k\to\infty}, \left(2\pi \sum_{u \in A_{r(R_k)} \backslash A_\rho} O(x^k,u) - \iint_{A_{r(R_k)} \backslash A_\rho} K^k d\omega^k \right) \\
&= \limsup_{k\to\infty} \int_{\partial(A_{r(R_k)} \backslash A_\rho)} \kappa_g^k ds^k \\
&\leq \limsup_{k\to\infty} \operatorname{curv}(\Gamma_{R_k}) - \int_{\partial D_\rho} \kappa_g ds,
\end{aligned}
$$

where $\operatorname{curv}(\Gamma_R)$ denotes the total absolute curvature of the curve Γ_R. It is however well known [J-M] that $\frac{1}{R}\Gamma_R$, Γ_R being the intersection of a finite total curvature end with a sphere of radius R, converges smoothly to a finite multiple of a great circle on the unit sphere. Hence $\operatorname{curv}(\Gamma_R)$ is bounded above independently of R and the assertion of the lemma follows.

We may now state and prove our first main result,

Theorem 3.5 *Let $\Gamma \subset B_1(0) \subset \mathbb{R}^n$ be a rectifiable Jordan curve and T a twodimensional subspace of $\mathbb{R}^n$ which is not orthogonal to the affine subspace of minimal dimension containing Γ. Then there exists a complete, non-compact, area-minimizing annulus M_∞ of finite total curvature spanning Γ and possessing a flat end asymptotic to T i. e. $M \backslash B_R(0)$ is a bounded graph over T for a sufficiently large R.*

Proof: We choose $M_0 = T$ in the construction of section 2 above. Let M_∞ be a limit surface of some sequence (M_{R_k}). It follows from the convex hull property of minimal surfaces that each M_R and hence M_∞ is contained in the slab $S := \{x \in \mathbb{R}^n \,|\, \operatorname{dist}(x,T) \leq 1\}$. Since by Proposition 3.4 M_∞ is known to be a properly immersed (possibly with branch points) finite total curvature surface it possesses a well defined asymptotic tangent space T_∞ $[H - O]$ and hence $\frac{1}{R} M_\infty \cap \partial B_R(0)$ converges smoothly to a multiple of the unit circle in T_∞ [J-M]. But M_∞ being contained in the slab S, this unit circle must be contained in T and, therefore, $T_\infty = T$. $M_0 = T$ being a plane, the area estimates Lemma 2.5 and, hence, Lemma 3.1 hold with $m = 1$. Now choose R so big that all points of $M_\infty \backslash B_R(0)$ are regular for the orthogonal projection $P : \mathbb{R}^n \to T$. This is possible since M_∞ has an immersed end by Proposition 3.4 and since T is the asymptotic tangent plane of M_∞. We may then conclude from the area estimate Lemma 3.1 (with $m = 1$) that the degree of $P|M_\infty \backslash B_R(0)$ must be ± 1, i. e. $M_\infty \backslash B_R(0)$ has only one sheet over T. $\qquad\square$

4 Flat ends of higher index

In this section we use the concept of incompressible surfaces to find, for suitable Jordan curves, solutions to the exterior Plateau problem with a flat end of index m, i. e. the minimal surface remains at bounded distance to its asymptotic tangent plane at infinity and the orthogonal projection of the surface onto this plane has the topological degree $m \in \mathbb{N}$. As in the preceding

sections, this solution is obtained as the limit of compact annuli where it is arranged that their projection map has the desired degree m. The main difficulty is of course to guarantee that in the limit the degree does not jump to a lower value. To accomplish this we make strong use of a barrier construction which enables us to apply the concept of incompressible surfaces. The barrier construction is made possible by our comparison principle for minimal submanifolds of arbitrary codimension [J-T].

Our barriers are the boundaries of the domains

$$D_\lambda = \{(z, w) \in \mathbb{R}^2 \times \mathbb{R}^{n-2} \,|\, |w| \leq \varphi_\lambda(|z|)\},$$

where φ_λ is one of the catenaries

$$\varphi_\lambda(t) = \lambda \ell n \left(t/\lambda + \sqrt{t^2/\lambda^2 - 1} \right), t \geq \lambda > 0.$$

It has already been shown in [J-T] that ∂D_λ is "2-mean convex" with respect to the normal pointing into D_λ and hence is a barrier for twodimensional minimal surfaces. We observe that $P(D_\lambda) = \{z \in \mathbb{R}^2 \,\big|\, |z| \geq \lambda\}$, where $P : \mathbb{R}^2 \times \mathbb{R}^{n-2} \to \mathbb{R}^2$ is the projection onto the first factor.

Let us consider a rectifiable Jordan curve $\Gamma \subset \text{int } D_\lambda$ such that $P(\Gamma)$ has the winding number $m \in \mathbb{N}$ with respect to the origin in the z-plane and for any $R > 1$ let Γ_R denote the Jordan curve

$$\Gamma_R(\zeta) = \left(\sqrt{R^2 - 1}\zeta^m, \zeta, 0, \ldots, 0 \right), \ \zeta \in S^1,$$

where we split $\mathbb{R}^n$ as $\mathbb{R}^n = \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{n-4}$.

We observe that $\Gamma_R \subset \partial B_R(0), \Gamma_R \subset \text{int } D_\lambda$ for sufficiently big R, $P(\Gamma_R)$ has winding number m, and, furthermore

$$\begin{aligned} \text{length}(\Gamma_R) &\leq mR, \\ \text{curv}(\Gamma_R) &\leq 2m\pi. \end{aligned}$$

It follows that, whenever $f : A_r \to D_\lambda$ is any continuous map such that $f|\partial A_r$ parametrizes $\Gamma - \Gamma_R$, then the induced map on fundamental groups $f_* : \pi_1(A_r) \to \pi_1(D_\lambda)$ is conjugate to the map $\mathbb{Z} \ni k \Rightarrow mk \in \mathbb{Z}$ and hence is injective. It follows from standard theory [T-T] that there exists a branched minimal annulus $M^R \subset D_\lambda$ spanning $\Gamma - \Gamma_R$, parametrized by a conformal, branched, harmonic map $X^R : A_{r(R)} \to D_\lambda$ in the above conjugacy class and which minimizes area among all such maps. One easily constructs admissible comparison maps of area not larger $mR^2\pi + mR\pi + a_0$ with a fixed constant a_0, independent of R and hence

$$\text{area}(M^R) \leq mR^2\pi + 2mR\pi + a_0$$

The condition of cohesion (cf. Corollary (2.4)), is trivially satisfied for M^R with $\varepsilon = 2m\pi\lambda$. From the above facts we may conclude that the theory of the preceding sections becomes applicable and that, for a suitable sequence $R_k \to \infty$, X^{R_k} converges locally uniformly to a

proper, branched, finite total curvature minimal surface $X : A \to D_\lambda$ spanning Γ. It follows immediately that the induced map X_* on fundamental groups is conjugate to $k \mapsto mk$, and, since $P : D_\lambda \to D_\lambda$ is homotopic to the identity of D_λ, and $(PX)_*$ is conjugate to $k \mapsto mk$ as well. This proves that PX has degree m on the unbounded component of $P(\Gamma)$. It follows from the convex hull property of minimal surfaces that all of the X^R and hence X remain at bounded distance from the z-plane. Hence we proved

Theorem 4.1 *Let $\Gamma \subset D_\lambda$ be a rectifiable Jordan curve such that $P\Gamma$ has the winding number m around the origin, where $P : \mathbb{R}^2 \times \mathbb{R}^{n-2} \to \mathbb{R}^2$ is the orthogonal projection. Then Γ spans a complete finite total curvature minimal annulus $M \subset D_\lambda$ with a flat end and the z-plane as its asymptotic tangent space. Moreover, $P|M$ has the topological degree m.*

5 Surfaces with a polynomial end

In this section we choose as our model surface M_0 the complex curve

$$(5.1) \qquad\qquad\qquad w = z^m$$

where $(z, w) \in \mathbb{C} \times \mathbb{C} \cong \mathbb{R}^4$ and $m \in \mathbb{N}$.

For some sufficiently large $R > 0$ we shall construct a neighborhood B of $M_0 \cap \{|w| > R\}$ which lies within bounded distance of M_0 and which has 2-mean convex boundary ∂B. Since B has the homotopy type of the circle, the methods of the previous paragraphs yield the existence of complete minimal annuli within B spanning a given homotopically nontrivial Jordan curve in B.

The asymptotical tangent plane at infinity of M_0 being the w-plane, we prefer, simultaneously interchanging z and w, to write (5.1) in the equivalent form

$$(5.2) \qquad\qquad\qquad w = z^{1/m},$$

thus representing the surface as an m-sheeted graph (outside of the origin) over its asymptotical tangent plane. Locally we may write (5.2) as a single valued graph

$$(5.3) \qquad\qquad\qquad z = x_1 + ix_2 \,,\; w = u_1 + iu_2$$

where u_1 and u_2 are functions of (x_1, x_2) which are solutions of the Cauchy- Riemann equations

$$D_1 u_1 = D_2 u_2,\; D_2 u_1 = -D_1 u_2,\; D_j = \frac{\partial}{\partial x_j}.$$

An orthogonal basis (e_1, e_2) of the tangent bundle and an orthonormal basis (N_1, N_2) of the normal bundle of M_0 are therefore given by

$$e_1 = (1, 0, D_1 u_1, D_1 u_2), e_2 = (0, 1, D_2 u_1, D_2 u_2)$$

and
$$N_1 = \lambda^{-1}(-D_1 u_1, D_2 u_2, 1, 0), N_2 = \lambda^{-1}(D_2 u_1, -D_2 u_2, 0, 1)$$

where $\lambda = \sqrt{1 + |Du_1|^2} = \sqrt{1 + |Du_2|^2}$. Since

$$|Du_j| = \frac{1}{m}\rho^{1/m-1}, |D^2 u_j| \leq c\rho^{1/m-2}$$

with $\rho = |z| = \sqrt{x_1^2 + x_2^2}$ we obtain

(5.4)
$$|D_j e_k| \leq c\rho^{1/m-2}, \ |\langle D_j e_k, e_\ell \rangle| \leq c\rho^{2/m-3}$$
$$|D_j N_k| \leq c\rho^{1/m-2}, |D_{ij} N_k| \leq c\rho^{1/m-3},$$

the constant c depending only on M_0.

As a first step we seek the desired neighborhood B of M_0 in the form

(5.5)
$$B = \{(z, w) \in \mathbb{C}^2| \quad \text{dist}((z, w), M_0) \leq f(|z|)\}$$

where f is still to be determined. In order to calculate the second fundamental form of ∂B we choose a local representation of M_0 of the form (5.3) which we denote by X. A local parametrization Y of ∂B is then obtained as

(5.6)
$$Y(x_1, x_2, N) = X(x_1, x_2) + f(\rho)N$$

where N varies in the normal unit sphere of M_0 at (x_1, x_2). The computations to follow are facilitated by means of

Lemma 5.1 *Given any point $p \in M_0$ we may locally modify the basis (N_1, N_2) in such a way that the relations*

(5.7)
$$\langle D_k N_i, N_j \rangle = 0 \qquad\qquad (i, j, k = 1, 2)$$

hold at p. The estimates (5.4) remain valid.

Proof: We define the modified basis by $\tilde{N}_1 = \cos\varphi N_1 + \sin\varphi N_2, \tilde{N}_2 = -\sin\varphi N_1 + \cos\varphi N_2$ with a function $\varphi = \varphi(x_1, x_2)$ yet to be determined. The new basis being again orthonormal, it suffices to consider the case $i = 1, j = 2$ in (5.7). One finds

$$\langle D_k \tilde{N}_1, \tilde{N}_2 \rangle = D_k \varphi + \langle D_k N_1, N_2 \rangle$$

and we may therefore choose φ as a first order polynomial in (x_1, x_2) such that $\varphi(p) = 0$ and $D_k \varphi(p) = -\langle D_1 N_1(p), N_2(p) \rangle$. $\qquad\square$

We start by calculating a local basis (T_1, T_2, T_3) for the tangent bundle of ∂B, substituting $N = \cos\theta N_1 + \sin\theta N_2$:

$$\begin{aligned}
T_j = D_j Y &= e_j + f'(\delta)D_j\rho(\cos\theta N_1 + \sin\rho N_2) + f(\cos\theta D_j N_1 + \sin\theta D_j N_2), \qquad (j = 1, 2),\\
T_3 = D_\theta Y &= f(-\sin\theta N_1 + \cos\theta N_2)
\end{aligned}$$

Shifting the angular variable θ we may assume that $\theta = 0$ at the point under consideration. The metric tensor (g_{ij}) then becomes

$$
\begin{aligned}
g_{ij} &= \lambda^2 \delta_{ij} + 2f \langle D_i N_1, e_j \rangle + f'^2 D_i \rho D_j \rho + f^2 \langle D_i N_1, D_j N_1 \rangle && (i, j = 1, 2), \\
g_{i3} &= 0 && (i = 1, 2) \\
g_{33} &= f^2.
\end{aligned}
$$

Introducing the normalizations

(5.8) $$ 0 \le f \le 1\,, \ |f'| \le 1, \ \ \rho \ge 1 $$

we obtain from (5.5) and (5.6)

$$ g_{ij} = \lambda^2 \delta_{ij} + O(\rho^{1/m-2} f + f'^2), \qquad (i, j = 1, 2) $$

Here and in what follows the growth rate of the O-term depends only on M_0, always taking (5.8) into account. For the inverse matrix we then get

(5.9)
$$
\begin{aligned}
g^{ij} &= \lambda^{-2} \delta_{ij} + O(\rho^{1/m-2} f + f'^2) && (i, j = 1, 2) \\
g^{i3} &= 0 && (i = 1, 2) \\
g^{33} &= f^{-2}.
\end{aligned}
$$

A normal vector ν for ∂B may be written in the form

$$ \nu = \alpha_1 e_1 + \alpha_2 e_2 + \cos \theta N_1 + \sin \theta N_2, $$

the coefficients α_j being determined from the relations

$$ 0 = \langle \nu, T_j \rangle = \lambda^2 \alpha_j + \sum_{i=1,2} f \alpha_i (\langle D_j N_1, e_i \rangle \cos \theta + \langle D_j N_2, e_i \rangle \sin \theta) + f' D_j \rho \qquad (j = 1, 2). $$

After an orthogonal coordinate transformation we may assume that

(5.10) $$ \langle D_i N_i, e_j \rangle = \kappa_i \langle e_i, e_j \rangle = \kappa_i \lambda^2 \delta_{ij}, $$

the κ_i being the principal curvatures of M_0 in direction N_1. Our normalization $\theta = 0$ and (5.10) lead to

$$ \alpha_j = -\lambda^{-2} (1 + f \kappa_j)^{-1} f' D_j \rho, $$

13

assuming from now on $\rho \geq \rho_0$ to ensure $1 + f\kappa_j \geq \frac{1}{2}$. Continuing towards the calculation of the second fundamental form we need

$$
\begin{aligned}
D_j T_i = {} & D_j e_i + (f''(\rho)D_i\rho D_j\rho + f'(\rho)D_{ij}\rho)(\cos\theta N_1 + \sin\theta N_2) + f'(\rho)D_i\rho(\cos\theta D_j N_1 + \\
& \sin\theta D_j N_2) + f'(\rho)D_j\rho(\cos\theta D_i N_1 + \sin\theta D_i N_2) + f(\rho)(\cos\theta D_{ij}N_1 + \\
& \sin\theta D_{ij}N_2), \hspace{6cm} (i,j=1,2),
\end{aligned}
$$

$$
D_3 T_i = D_\theta T_i = f'(\rho)D_i\rho(-\sin\theta N_1 + \cos\theta N_2) + f(\rho)(-\sin\theta D_i N_1 + \cos\theta D_i N_2), \quad (i=1,2),
$$

$$
D_3 T_3 = D_\theta T_3 = f(-\cos\theta N_1 - \sin\theta N_2).
$$

Again using (5.7), (5.10) and the normalization $\theta = 0$ we calculate $b_{ij} := -\langle \nu, D_j T_i \rangle$ as follows:

$$
\begin{aligned}
b_{ij} = {} & -\sum_{k=1,2} \alpha_k \langle D_j e_i, e_k \rangle - f'(\rho)D_i\rho\,\alpha_j\kappa_j\lambda^2 - f'(\rho)D_j\rho\,\alpha_i\kappa_i\lambda^2 + \lambda^2\kappa_i\delta_{ij} \\
& -f''(\rho)D_i\rho D_j\rho - f'(\rho)D_{ij}\rho - f\langle \sum \alpha_k e_k + N_1, D_{ij}N_1 \rangle, \hspace{2cm} (i,j=1,2),
\end{aligned}
$$

$$
b_{i3} = -\sum_{k=1,2} \alpha_k f\langle D_i N_2, e_k \rangle \hspace{7cm} (i=1,2),
$$

$$
b_{33} = f.
$$

By means of (5.4), and (5.10) we obtain the estimates

$$
\begin{aligned}
(5.11) \qquad b_{ij} &= \lambda^2\kappa_i\delta ij - (f''(\rho)D_i\rho D_j\rho + f'D_{ij}\rho) \\
&\quad + O(\rho^{2/m-3}f' + \rho^{1/m-2}f'^2 + \rho^{2/m-4}), \hspace{2cm} (i,j=1,2), \\
b_{i3} &= O(\rho^{1/m-2}f'f) \hspace{6cm} (i=1,2), \\
b_{33} &= f.
\end{aligned}
$$

Let us now consider the submatrices

$$
G = (g^{ij})_{ij=1,2},\, B = (b_{ij})_{i,j=1,2}.
$$

(5.19) and (5.11) yield the following representation for the second fundamental tensor A in direction ν:

$$
(5.12) \qquad A = (g^{ij})(b_{ij}) = A_0 + O(\rho^{1/m-2}(ff' + f'/f))
$$

with

$$A_0 = \begin{pmatrix} & & 0 \\ G & B & \\ & & 0 \\ 0 & 0 & 1/f \end{pmatrix}.$$

Since $G = \lambda^{-2}I + O(f'^2 + \rho^{1/m-2})$ and $B = O(\rho^{1/m-2} + |f''| + \rho^{-1}|f'|)$ we conclude that

$$(5.13) \qquad GB = \lambda^{-2}B + O((f'^2 + \rho^{1/m-2})(\rho^{1/m-2} + |f''| + |f'|/\rho)).$$

Let us now choose

$$f(\rho) = 1 - \rho^{-\beta}$$

with $\beta \in]0, 2[$. Using the minimality of M_0 we obtain

$$\text{trace } B = -(f''(\rho) + \frac{1}{\rho}f'(\rho)) + O(\rho^{2/m-3}f' + \rho^{1/m-2}f'^2 + \rho^{2/m-4}) = \rho^{-2-\beta} + O(\rho^{2/m-4}).$$

Together with (5.15) this yields the estimate for the eigenvalues $\lambda_i(GB)$

$$(5.14) \qquad \begin{aligned} \lambda_1(GB) + \lambda_2(GB) &= \text{trace}(GB) \\ &= \lambda^{-2}\rho^{-2-\beta} + O(\rho^{2/m-4}). \end{aligned}$$

The eigenvalues $\lambda_i(A_0)$ being

$$\lambda_1(A_0) = \lambda_1(GB), \lambda_2(A_0) = \lambda_2(GB), \lambda_3(A_0) = 1/f,$$

where $\lambda_i(A_0) = O(\|B\|) = O(\rho^{1/m-2})$ for $i = 1, 2$, we see immediately that

$$(5.15) \qquad \begin{aligned} \lambda_1(A_0) + \lambda_2(A_0) &\geq \frac{1}{2}\lambda^{-2}\rho^{-2-\beta}, \\ \lambda_i(A_0) + \lambda_3(A_0) &\geq \frac{1}{2}f(\rho)^{-1}. \end{aligned}$$

provided that we choose $\beta \in]0, 2 - 2/m[$ and $\rho \geq \rho_0, \rho_0$ depending on M_0 and β. It remains to estimate the difference between the eigenvalues $\lambda_i(A)$ and $\lambda_i(A_0)$. We have $f'(\rho)/f(\rho) \leq 2\beta\rho^{-1-\beta}$ for $\rho \geq 2^{1/\beta}$ and hence we conclude from (5.12) that

$$(5.16) \qquad \lambda_i(A) - \lambda_i(A_0) = O(\rho^{1/m-3-\beta}).$$

15

Comparing (5.15) and (5.16) we see that the sum of any two eigenvalues of A is positive for $\rho \geq \rho_0$, proving the 2-mean convexity of ∂B outside of the cylinder $x_1^2 + x_2^2 \leq \rho_0^2$. Let us remark that all our estimates being direct, the bound ρ_0 can be computed explicitly.

Since we can show the 2-mean convexity of the boundary of the set B defined in (5.5) only for $|z| \geq \rho_0$ for some sufficiently large ρ_0 it becomes necessary to modify the definition of B in order to obtain a set with everywhere 2-mean convex boundary. Consequently we define

(5.17) $$B = B_\rho \cup T_\rho$$

with

(5.18) $$B_\rho = \{(z, w) \,|\, \text{dist}((z, w), M_0) \leq f(|z|), |z| \geq \rho\}$$

and T_ρ essentially being a tubular neighborhood of the curve $C_\rho = M_0 \cap \{|z| = \rho\}$
Introducing polar coordinates (ρ, φ) in the z-plane and a further angular variable ψ we parameterize ∂T_ρ by the map

(5.19) $$Z_\rho(s, N, \psi) = X(\rho, \varphi) + \delta g(\psi) N + h(s) \sin \psi D_\rho X(\rho, \varphi)$$

where $s = \rho\varphi, 0 \leq \varphi \leq 2m\pi, \pi/2 \leq \psi \leq \pi/2$, N varies in the normal unit sphere bundle of M_0 restricted to $C_\rho, \delta > 0$, and g as well as h are yet to be determined with $g(\psi) = \cos\psi$ near $\psi = \pm\pi/2$. Let us remark that $\psi = \pm\pi/2$ does not introduce any boundary in (5.19) since the curve $Z_\rho(s, N, \cdot)$ closes up withe $Z_\rho(s, -N, \cdot)$. The plan is to glue half of ∂T_ρ, i. e. the portion of ∂T_ρ corresponding to $\psi \in [-\pi/2, 0]$ to 5.14 ∂B_ρ along the surface $Z_\rho(\cdot, \cdot, 0)$ for some large $\rho = \rho_0, B_\rho$ being defined in (5.18). The gluing condition of order zero is

$$Z_\rho(\rho\varphi, N, 0) = Y(\rho, \varphi, N),$$

which is equivalent to

(5.20) $$\delta g(0) = f(\rho).$$

It will be sufficient to make the gluing of class C^1, simultaneously keeping the pieces ∂B_ρ and ∂T_ρ strictly 2-mean convex, because then $\partial B_\rho \cup \partial T_\rho$ may be further smoothened in a standard way.

Let us start by computing a local basis (V_1, V_2, V_3) of the tangent bundle of ∂T_ρ:

$$V_1 := D_s Z_\rho = \tfrac{1}{\rho} D_\varphi Z_\rho = \tfrac{1}{\rho}\{D_\varphi X + \delta g(\psi)(\cos\theta D_\varphi N_1 + \sin\theta D_\varphi N_2)$$

(5.21) $$+ \rho h'(s) \sin\psi D_\rho X + h \sin\psi D_{\rho\varphi} X\}$$

16

$$V_2 := D_\theta Z_\rho = \delta g(\psi)(-\sin\theta N_1 + \cos\theta N_2)$$

$$V_3 := D_\psi Z_\rho = \delta g'(\psi)(\cos\theta N_1 + \sin\theta N_2) + h\cos\psi D_\rho X$$

Comparing with (5.6) at $\theta = 0, \psi = 0$ and, besides (5.20) also requiring that

(5.22)
$$\delta g'(0) = f'(\rho)$$

we see that

$$D_\varphi Y = \rho V_1, \; D_\theta Y = V_2 ,$$

$$D_\rho Y = D_\rho X + f' N_1 + f D_\rho N_1,$$

$$V_3 = h D_\rho X + f' N_1.$$

Hence, in order that the tangent spaces of ∂B_ρ and ∂T_ρ along the glueing surface coincide we must adjust h in such a way that $D_\rho Y$ becomes a linear combination of V_1, V_2, and V_3. According to Lemma 5.1 we may write

$$D_\rho N_1 = \alpha_1 D_\rho X + \alpha_2 D_\varphi X, \; D_\varphi N_1 = \beta_1 D_\rho X + \beta_2 D_\varphi X$$

with

$$\alpha_1 = |D_\rho X|^{-2}\langle D_\rho N_1, D_\rho X\rangle, \alpha_2 = |D_\varphi X|^{-2}\langle D_\rho N_1, D_\varphi X\rangle$$

$$\beta_1 = |D_\rho X|^{-2}\langle D_\varphi N_1, D_\rho X\rangle, \; \beta_2 = |D_\varphi X|^{-2}\langle D_\varphi N_1, D_\varphi X\rangle.$$

From this we obtain

$$D_\varphi X = (1 + \beta_2 f)^{-1} V_1 - f\beta_1(1 + \beta_2 f)^{-1} D_\rho X,$$

$$D_\rho N_1 = (\alpha_1 - f\beta_1(1 + \beta_2 f)^{-1})D_\rho X + \alpha_2(1 + \beta_2 f)^{-1} V_1,$$

and hence

$$D_\rho Y = (1 + f(\alpha_1 - f\beta_1(1 + \beta_2 f)^{-1})D_\rho X + f' N_1 + f\alpha_2(1 + \beta_2 f)^{-1} V_1,$$

from which we see that we must choose

$$h = 1 + f(\rho)(\alpha_1 - f(\rho)\beta_1(1 + \beta_2 f(\rho))^{-1}),$$

α_j, β_j being functions of the angle φ.

Using

$$|D_\rho X|^2 = 1 + \tfrac{1}{m^2}\rho^{2/m-2}, \ |D_\varphi X|^2 = \rho^2 |D_\rho X|^2,$$

(5.23)
$$|D_{\rho\rho} X|^2 = \tfrac{(m-1)^2}{m^4}\rho^{2/m-4}, \ |D_{\varphi\varphi} X|^2 = \rho^2(1 + \tfrac{1}{m^4}\rho^{2/m-2}),$$

$$|D_{\rho\varphi} X|^2 = 1 + \tfrac{1}{m^4}\rho^{2/m-2},$$

$$|D_\rho N_j| = O(\rho^{1/m-2}), |D_\varphi N_j| = O(\rho^{1/m-1})$$

one verifies that α_j and β_j as well as their derivatives with respect to φ are at most of the order $O(\rho^{1/m-1})$ and hence, keeping $s = \rho\varphi$ in mind,

(5.24)
$$h = 1 + O(\rho^{1/m-1}), h' = O(\rho^{1/m-2})$$
$$h'' = O(\rho^{1/m-3}).$$

Let us now define the function g. We choose $\eta \in C^\infty(\mathbb{R})$ with $|\eta| \leq 1$, $\eta(\psi) = 0$ for $\psi \leq -\pi/4$, $\eta(0) = 0$, $\eta'(0) = 1$ and, remembering that $f(\rho) = 1 - \rho^{-\beta}$, we define

$$g(\psi) = \cos\psi + \beta\rho^{-1-\beta}(1 - \rho^{-\beta})^{-1}\eta(\psi).$$

Setting $\delta := 1 - \rho^{-\beta}$ we see that (5.20) and (5.22) are fulfilled. Moreover,

(5.25)
$$D_\psi^j(g(\psi) - \cos\psi) = O(\rho^{-1-\beta}) \text{ for } j = 0, 1, 2.$$

We would like to argue that, for large ρ, (5.19) is a C^2-small perturbation of the standard round tubular neighborhood of radius 1 of the curve $X(\rho, \cdot)$ where $\delta = 1$ and $h = |D_\rho X|^{-1} = 1/\sqrt{1 + \tfrac{1}{m^2}\rho^{2/m-2}}$, implying that (5.19) would be 2-mean convex for sufficiently large ρ, since this is certainly the case for the standard tube if ρ is sufficiently large. The C^2 closeness is obvious for the mapping Z_ρ on account of (5.24) and (5.25). In order to conclude the closeness of the second fundamental forms one must make sure that the first fundamental forms remain uniformly nonsingular. This does not seem to be case near $\psi = -\pi/2$ where V_2 in (5.23) becomes zero. But this this is simply a deficiency of our coordinate system which disappears if we introduce new coordinates in the range $[-\tfrac{\pi}{2}, \tfrac{\pi}{4}]$ according to

$$x_1 = \cos\psi\cos\theta, \ x_2 = \cos\psi\sin\theta$$

transforming (5.19) into

$$Z_\rho = X(\rho, \varphi) + \delta(x_1 N_1 + x_2 N_2) - h(s)\sqrt{1 - (x_1^2 + x_2^2)} D_\rho X(\rho, \varphi).$$

It is easily seen that the metric tensor of Z_ρ in the coordinates $(s = \rho\varphi, x_1, x_2)$ is uniformly nonsingular for large ρ in the range $x_1^2 + x_2^2 \leq 1/2$. Hence we proved

18

Theorem 5.2 *There is an explicitly computable number $\rho_0 > 0$ so that the surface*

$$M_{\rho_0} := \{(z, w) \in \mathbb{C}^2 \,|\, w = z^m, |w| \geq \rho_0\}$$

admits a neighborhood B with 2-mean convex boundary. B retracts onto M_{ρ_0} and the points of B are within distance 1 from M_{ρ_0}, the distance between ∂B and M_{ρ_0} approaching 1 at infinity.

$\square$

We remark that the distance 1 condition is just a normalization; the number 1 can be replaced by any positiv number, however with the effect that ρ_0 has to be adjusted. On the basis of Theorem 5.2 we finally obtain the following existence theorem along the same lines as in section 4.

Theorem 5.3 *Let M_{ρ_0} and B as in Theorem 5.2. Then, given any rectifiable Jordan curve Γ in B which is freely homotopic in B to ∂M_{ρ_0}, there exists a complete annulus type minimal surface M in B with boundary Γ and finite total curvature end. In particular, M stays within distance 1 from M_{ρ_0} and hence M behaves like $w = z^m$ at infinity.*

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